Last update: 1:17am, Feb 27.

Math 565: Functional Analysis | HOMEWORK 4 | **Due: Mar 10, 23:59**

1. (a) Prove that every finite-dimensional normed vector space $(X, \|\cdot\|)$ is isomorphic to $(\mathbb{C}^n, \|\cdot\|_1) =: \ell^1(n)$ for some $n \in \mathbb{N}$ as normed vector spaces, i.e. there is a vector space isomorphism $\varphi : X \to \mathbb{C}^n$ so that φ, φ^{-1} are continuous (equivalently, bounded).

REMARK: Recall that all the ℓ^p norms on $\mathbb{C}^n$ are Lipschitz equivalent to each other for all $1 \leqslant p \leqslant \infty$. I just chose the ℓ^1 norm for convenience.

HINT: Fix a linear basis $x_0, x_1, \dots, x_{n-1}$ for X of normal (norm 1) vectors and set $\varphi(x_i) := e_i$, so immediately $\|x\| \leqslant \|\varphi(x)\|_1$ for all $x \in X$. For the other inequality, let X_i denote the span of $\{x_0, x_1, ..., x_{n-1}\} \setminus \{x_i\}$ and suppose by induction that X_i is isomorphic to $\mathbb{C}^{n-1}$, and is hence closed in X (why?), so $\delta_i := \|x_i + X_i\| = \operatorname{dist}(x_i, X_i) > 0$. Realize that $\|x\| \geqslant |\alpha_i|\delta_i$ for each $x = \sum_{i<n} \alpha_i x_i$ and $i < n$.

(b) Deduce that every finite dimensional subspace of an infinite dimensional normed vector space is closed and nowhere dense.

(c) Finally, conclude that an infinite dimensional Banach space does not have a countable linear (Hamel) basis. In particular, the space $\mathbb{C}[t]$ of polynomials is not a Banach space under any norm.

2. [*Optional*] Prove that there is a dense G_δ, hence comeagre, subset of $\mathbb{R}$ which is Lebesgue null. Prove more generally that a μ-null dense G_δ subset exists in every separable topological space X for every outer regular atomless Borel measure μ.

3. Prove that lcH spaces are Baire.

4. [*Optional*] Follow the steps below to show that a generic function in $C([0,1])$ is nowhere differentiable, i.e. the set of nowhere differential functions is comeagre in $C([0,1])$.

(i) Prove that given $m \in \mathbb{N}$, any function $f \in C([0,1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0,1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geqslant m$.

(ii) For each $n \geqslant 1$, let E_n be the set of all functions $f \in C([0,1])$, for which there is $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leqslant n|x - x_0|$ for all $x \in [0,1]$. Show that E_n is nowhere dense using the fact that if g is as in (i) with $m = 2n$, then some open neighbourhood of g is disjoint from E_n.

5. Let X, Y be normed vector spaces and $T \in B(X, Y)$, so $\ker T$ is a closed subspace of X. Let $\pi : X \to X/\ker T$ be the quotient map. Prove:

(a) T uniquely factors as $T = \widetilde{T} \circ \pi$, where $\widetilde{T} \in B(X/\ker T, Y)$ and $\|\widetilde{T}\| = \|T\|$.

HINT: For $\|\widetilde{T}\| = \|T\|$, use that for each $\varepsilon > 0$, every coset $C \in X/\ker T$ with $\|C\| = 1$ admits a representative $x \in C$ with $1 \leqslant \|x\| \leqslant 1 + \varepsilon$.

(b) **First Isomorphism Theorem for Banach spaces.** Let X, Y be Banach spaces and $T \in B(X, Y)$. Then $T/\ker T$ is isomorphic[1] to $T(X)$ if and only if $T(X)$ is closed.

6. Follow the steps below to prove that every separable Banach space X is isomorphic to a quotient of $\ell^1 := \ell^1(\mathbb{N})$ by a closed subspace.

(i) Let $D \subseteq X$ be a countable dense subset of the closed unit ball B of X. Without loss of generality, we may assume that D is closed under doubling, i.e. if $x \in D$ and $2x \in B$, then $2x \in D$. Enumerate $D = (x_n)_{n\in\mathbb{N}}$ and define $T : \ell^1 \to X$ by $f \mapsto \sum_{n\in\mathbb{N}} f(n)x_n$. Show that T is well-defined (i.e. the series $\sum_{n\in\mathbb{N}} f(n)x_n$ converge) and bounded.

(ii) Prove that T is surjective.

HINT: Use the construction as in the last claim in the proof of the open mapping theorem to write $x = \sum_{k\in\mathbb{N}} x_{n_k}$ such that $2^k x_{n_k} \in B$ for each $k \in \mathbb{N}$.

(iii) Deduce that $\ell^1/\ker T$ is isomorphic to X.

7. There is no slowest rate of decay of the terms of an absolutely convergent series. More precisely, there is no sequence $(a_n) \in \ell^1$ of positive reals such that for each $(x_n) \in \ell^1$, the sequence (x_n/a_n) is bounded.

HINT: If such an (a_n) exists, then $T : \ell^\infty \to \ell^1$ defined by $Tf(n) := a_n f(n)$ is an isomorphism.

8. Let $Y := C([0, 1])$ and $X := C^1([0, 1])$, both equipped with the uniform norm. Show that the differentiation map $D : X \to Y : f \mapsto f'$ is closed but not bounded. Why is this happening?

9. Let X, Y be normed vector spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

HINT: Define a family $\mathcal{F} \subseteq Y^{**}$ such that the boundedness of $f \circ T$ is equivalent to $\sup_{S\in\mathcal{F}} \|Sf\| < \infty$.

CAUTION: Folland assumes that X, Y are Banach spaces, but I do not see where this assumption would be used. Please let me know if you think we need this assumption.

10. Let X be a topological vector space whose topology is generated by a countable family $\{p_n\}_{n\in\mathbb{N}}$ of seminorms. Prove that X is metrizable by the translation-invariant metric

$$d(x, y) := \sum_{n\in\mathbb{N}} 2^{-n}\, \widetilde{p}_n(x - y),$$

where $\widetilde{p}_n(z) := \min\{p_n(z), 1\}$ for $z \in X$.

[1] Recall that an isomorphism of normed vector spaces is a linear homeomorphism, not necessarily an isometry.